WHEN IS GROUP COHOMOLOGY FINITARY?

MARTIN HAMILTON

ABSTRACT. If G is a group, then we say that the functor $H^n(G,-)$ is *finitary* if it commutes with all filtered colimit systems of coefficient modules. We investigate groups with *cohomology almost everywhere finitary*; that is, groups with nth cohomology functors finitary for all sufficiently large n. We establish sufficient conditions for a group G possessing a finite dimensional model for $\underline{E}G$ to have cohomology almost everywhere finitary. We also prove a stronger result for the subclass of groups of finite virtual cohomological dimension, and use this to answer a question of Leary and Nucinkis. Finally, we show that if G is a locally (polycyclic-by-finite) group, then G has cohomology almost everywhere finitary if and only if G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup of G is finitely generated.

1. Introduction

Let G be a group and $n \in \mathbb{N}$. The nth cohomology of G is a functor

$$H^n(G,-) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},-)$$

from the category of $\mathbb{Z}G$-modules to the category of $\mathbb{Z}$-modules, and we say that it is *finitary* if it commutes with all filtered colimit systems of coefficient modules (see §3.18 in [1]; also §6.5 in [14]).

Brown [4] has characterised groups of type FP_∞ in terms of finitary functors (see also results of Bieri, Theorem 1.3 in [2]):

Proposition 1.1. *A group G is of type FP_∞ if and only if $H^n(G,-)$ is finitary for all n.*

It seems natural, therefore, to consider groups whose nth cohomology functors are finitary for *almost all n*. We say that such a group has *cohomology almost everywhere finitary.*

In this paper, we shall investigate groups with cohomology almost everywhere finitary. We begin with the class of locally (polycyclic-by-finite) groups, and in §3 we prove the following:

2000 *Mathematics Subject Classification.* 20J06 20J05 18G15.

Key words and phrases. cohomology of groups, finitary functors.

Theorem A. *Let G be a locally (polycyclic-by-finite) group. Then G has cohomology almost everywhere finitary if and only if G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup of G is finitely generated.*

If G is a locally (polycyclic-by-finite) group with cohomology almost everywhere finitary, then a result of Kropholler (Theorem 2.1 in [10]) shows that G has a finite dimensional model for the classifying space $\underline{E}G$ for proper actions and, furthermore, that there is a bound on the orders of the finite subgroups of G (see §5 of [13] for a brief explanation of the classifying space $\underline{E}G$). In §9 we prove the following Lemma:

Lemma 1.2. *Let G be a locally (polycyclic-by-finite) group. Then the following are equivalent:*

(i) *There is a finite dimensional model for $\underline{E}G$, and there is a bound on the orders of the finite subgroups of G;*
(ii) *G has finite virtual cohomological dimension; and*
(iii) *There is a finite dimensional model for $\underline{E}G$, and G has finitely many conjugacy classes of finite subgroups.*

Therefore, we can reduce our study to those groups G which have finite virtual cohomological dimension. We then have the following short exact sequence:

$$N \rightarrowtail G \twoheadrightarrow Q,$$

where N is a torsion-free, locally (polycyclic-by-finite) group of finite cohomological dimension, and Q is a finite group. Hence, in order to prove Theorem A, we must consider three cases. The first case is when G is torsion-free. In this case, G has finite cohomological dimension, so $H^n(G,-)=0$, and hence is finitary, for all sufficiently large n. The next simplest case, when G is the direct product $N \times Q$ is treated in §2, and the general case is then proved in §3.

Now, if G is any group with cohomology almost everywhere finitary, and H is a subgroup of G of finite index, then it is always true that H also has cohomology almost everywhere finitary (see Lemma 2.1 below). However, in the case of locally (polycyclic-by-finite) groups we can say much more than this:

Corollary B. *Let G be a locally (polycyclic-by-finite) group. If G has cohomology almost everywhere finitary, then every subgroup of G also has cohomology almost everywhere finitary.*

This is not true in general, however, as can been seen from Proposition 4.1 below.

Next, we consider the class of elementary amenable groups, and prove the following in §5:

Proposition C. *Let G be an elementary amenable group with cohomology almost everywhere finitary. Then G has finitely many conjugacy classes of finite subgroups, and $C_G(E)$ is finitely generated for every $E \leq G$ of order p.*

In §6 we investigate the class of groups of finite virtual cohomological dimension. In this section we work over a ring R of prime characteristic p, instead of over $\mathbb{Z}$, by defining the nth cohomology of a group G as

$$H^n(G, -) := \operatorname{Ext}^n_{RG}(R, -).$$

In order to make it clear that we are now working over R, we say that $H^n(G, -)$ is *finitary over R* if and only if the functor $\operatorname{Ext}^n_{RG}(R, -)$ is finitary. We have an analogue of Proposition 1.1, characterising the groups of type FP_∞ over R as those with nth cohomology functors finitary over R for all n. We can similarly define the notion of a group having *cohomology almost everywhere finitary over R*, and we prove the following result:

Theorem D. *Let G be a group of finite virtual cohomological dimension, and R be a ring of prime characteristic p. Then the following are equivalent:*

(i) *G has cohomology almost everywhere finitary over R;*
(ii) *G has finitely many conjugacy classes of elementary abelian p-subgroups and the normalizer of every non-trivial elementary abelian p-subgroup of G is of type FP_∞ over R; and*
(iii) *G has finitely many conjugacy classes of elementary abelian p-subgroups and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over R.*

We then adapt the proof of Theorem D slightly, and in §7 we use it to prove the following result, which answers a question of Leary and Nucinkis (Question 1 in [13]):

Theorem E. *Let G be a group of type VFP over $\mathbb{F}_p$, and P be a p-subgroup of G. Then the centralizer $C_G(P)$ of P is also of type VFP over $\mathbb{F}_p$.*

Finally, in §8 we return to working over $\mathbb{Z}$, and consider the class of groups which possess a finite dimensional model for $\underline{E}G$. We prove the following:

Proposition F. *Let G be a group which possesses a finite dimensional model for the classifying space $\underline{E}G$ for proper actions. If*

(i) *G has finitely many conjugacy classes of finite subgroups; and*

(ii) *The normalizer of every non-trivial finite subgroup of G has cohomology almost everywhere finitary,*

Then G has cohomology almost everywhere finitary.

However, the converse of this result is false, and we shall exhibit counter-examples in §8 by using a theorem of Leary (Theorem 20 in [12]). These counter-examples show that the converse of Proposition F is false even for the subclass of groups of finite virtual cohomological dimension.

1.1. **Acknowledgements.** I would like to thank my research supervisor Peter Kropholler for all of his advice and support throughout this project. I would also like to thank Ian Leary for useful discussions concerning the converse of Proposition F, and for suggesting that my results could be used to prove Theorem E.

2. The Direct Product Case of Theorem A

Suppose that $G = N \times Q$, where N is a torsion-free, locally (polycyclic-by-finite) group of finite cohomological dimension, and Q is a non-trivial finite group. We wish to show that G has cohomology almost everywhere finitary if and only if the normalizer of every non-trivial finite subgroup of G is finitely generated. Now, if F is a non-trivial finite subgroup of G, then F must be a subgroup of Q, and so N is a subgroup of $N_G(F)$ of finite index. Hence, $N_G(F)$ is finitely generated if and only if N is. It is therefore enough to prove that G has cohomology almost everywhere finitary if and only if N is finitely generated.

We begin by assuming that N is finitely generated. Therefore N is polycyclic-by-finite, and hence of type FP_∞ (Examples 2.6 in [2]). The property of type FP_∞ is inherited by supergroups of finite index, so G is also of type FP_∞. Therefore, by Proposition 1.1, we see that G has cohomology almost everywhere finitary.

For the converse, we shall prove a more general result which does not place any restrictions on the group N. Firstly, we need the following three lemmas:

Lemma 2.1. *Let G be a group, and H be a subgroup of finite index. If $H^n(G, -)$ is finitary, then $H^n(H, -)$ is also finitary.*

Proof. Suppose that $H^n(G, -)$ is finitary. From Shapiro's Lemma (Proposition 6.2 §III in [5]), we have:

$$H^n(H, -) \cong H^n(G, \mathrm{Coind}_H^G -).$$

Then, as H has finite index in G, it follows from Lemma 6.3.4 in [19] that $\mathrm{Coind}_H^G(-) \cong \mathrm{Ind}_H^G(-)$. Therefore,

$$H^n(H,-) \cong H^n(G, \mathrm{Ind}_H^G -) \cong H^n(G, - \otimes_{\mathbb{Z}H} \mathbb{Z}G),$$

and as tensor products commute with filtered colimits, we see that $H^n(H,-)$ is the composite of two finitary functors, and hence is itself finitary. $\square$

Lemma 2.2. *Let G be a group, and $R_1 \to R_2$ be a ring homomorphism. If $H^n(G,-)$ is finitary over R_1, then $H^n(G,-)$ is finitary over R_2.*

Proof. We see from Chapter 0 of [2] that for any R_2G-module M we have the following isomorphism:

$$\mathrm{Ext}^n_{R_2G}(R_2, M) \cong \mathrm{Ext}^n_{R_1G}(R_1, M),$$

where M is viewed as an R_1G-module via the homomorphism $R_1 \to R_2$. The result now follows. $\square$

Lemma 2.3. *Let $F_1, F_2 : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$, and suppose that F is the direct sum of F_1 and F_2. If F is finitary, then so are F_1 and F_2.*

Proof. As F is the direct sum of F_1 and F_2, we have the following exact sequence of functors:

$$0 \to F_1 \to F \to F_2 \to 0.$$

Let (M_λ) be a filtered colimit system of R-modules. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\varinjlim_\lambda F_1(M_\lambda) \arrow[r, tail] \arrow[d, "f_1"'] & \varinjlim_\lambda F(M_\lambda) \arrow[r, two heads] \arrow[d, "f"] & \varinjlim_\lambda F_2(M_\lambda) \arrow[d, "f_2"] \\
F_1(\varinjlim_\lambda M_\lambda) \arrow[r, tail] & F(\varinjlim_\lambda M_\lambda) \arrow[r, two heads] & F_2(\varinjlim_\lambda M_\lambda)
\end{tikzcd}$$

As F is finitary, we see that the map f is an isomorphism. It then follows from the Snake Lemma that f_1 is a monomorphism and f_2 is an epimorphism.

Now, as F is the direct sum of F_1 and F_2, we also have the following exact sequence of functors:

$$0 \to F_2 \to F \to F_1 \to 0,$$

and hence the following commutative diagram with exact rows:

$$\begin{tikzcd}
\varinjlim_\lambda F_2(M_\lambda) \arrow[r, tail] \arrow[d, "f_2"'] & \varinjlim_\lambda F(M_\lambda) \arrow[r, two heads] \arrow[d, "f"] & \varinjlim_\lambda F_1(M_\lambda) \arrow[d, "f_1"] \\
F_2(\varinjlim_\lambda M_\lambda) \arrow[r, tail] & F(\varinjlim_\lambda M_\lambda) \arrow[r, two heads] & F_1(\varinjlim_\lambda M_\lambda)
\end{tikzcd}$$

and a similar argument to above shows that f_2 is a monomorphism and f_1 is an epimorphism. The result now follows. $\square$

Proposition 2.4. *Let Q be a non-trivial finite group, and N be any group. If there is some natural number k such that $H^k(N \times Q, -)$ is finitary, then N is finitely generated.*

Proof. Suppose that $H^k(N \times Q, -)$ is finitary. As Q is a non-trivial finite group, we can choose a subgroup E of Q of order p, for some prime p, so $N \times E$ is a subgroup of $N \times Q$ of finite index. It then follows from Lemma 2.1 that $H^k(N \times E, -)$ is also finitary. Then, by Lemma 2.2, we see that $H^k(N \times E, -)$ is finitary over $\mathbb{F}_p$.

Let M be any $\mathbb{F}_p N$-module, and $\mathbb{F}_p$ be the trivial $\mathbb{F}_p E$-module. Applying the Künneth Theorem gives the following isomorphism:

$$\begin{aligned} H^k(N \times E, M) &\cong \textstyle\bigoplus_{i+j=k} H^i(N, M) \otimes_{\mathbb{F}_p} H^j(E, \mathbb{F}_p) \\ &\cong \textstyle\bigoplus_{i=0}^{k} H^i(N, M), \end{aligned}$$

and as this holds for any $\mathbb{F}_p N$-module M, we have an isomorphism of functors for modules on which E acts trivially. Then, as $H^k(N \times E, -)$ is finitary over $\mathbb{F}_p$, it follows from Lemma 2.3 that $H^0(N, -)$ is also finitary over $\mathbb{F}_p$. It then follows that N is finitely generated (see, for example, Proposition 2.1 in [2]).

$\square$

The converse of the direct product case now follows immediately.

3. Proof of Theorem A

Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. We begin with the following useful result of Cornick and Kropholler (Theorem A in [7]):

Proposition 3.1. *Let G be a group possessing a finite dimensional model for $\underline{E}G$, and M be an RG-module. Then M has finite projective dimension over RG if and only if M has finite projective dimension over RH for all finite subgroups H of G.*

Theorem 3.2. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. If G has cohomology almost everywhere finitary, then the normalizer of every non-trivial finite subgroup of G is finitely generated.*

Proof. Let F be a non-trivial finite subgroup of G, so we can choose a subgroup E of F of order p, for some prime p. As G has finite virtual cohomological dimension, it has a torsion-free normal subgroup N of

finite index. Let $H := NE$, so it follows from Lemma 2.1 that H has cohomology almost everywhere finitary.

Let Λ denote the set of non-trivial finite subgroups of H, so Λ consists of subgroups of order p. Now H acts on this set by conjugation, so the stabilizer of any $K \in \Lambda$ is $N_H(K)$. Also, for each $K \in \Lambda$, we see that the set of K-fixed points Λ^K is simply the set $\{K\}$, because if K fixed some $K' \neq K$, then KK' would be a subgroup of H of order p^2, which is a contradiction.

We have the following short exact sequence:

$$J \rightarrowtail \mathbb{Z}\Lambda \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z},$$

where ε denotes the augmentation map. For each $K \in \Lambda$, we see that J is free as a $\mathbb{Z}K$-module with basis $\{K' - K : K' \in \Lambda\}$. Now, as H has finite virtual cohomological dimension, it has a finite dimensional model for $\underline{E}H$ (Exercise §VIII.3 in [5]), and so it follows from Proposition 3.1 that J has finite projective dimension over $\mathbb{Z}H$. Now, the short exact sequence $J \rightarrowtail \mathbb{Z}\Lambda \twoheadrightarrow \mathbb{Z}$ gives rise to a long exact sequence in cohomology, and as J has finite projective dimension, we conclude that for all sufficiently large n we have the following isomorphism:

$$H^n(H, -) \cong \mathrm{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}\Lambda, -).$$

Next, as H acts on Λ, we can split Λ up into its H-orbits, so

$$\Lambda = \coprod_{K \in \mathscr{C}} H_K \backslash H = \coprod_{K \in \mathscr{C}} N_H(K) \backslash H,$$

where K runs through a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial finite subgroups of H. This gives the following isomorphism:

$$\begin{array}{rcl} H^n(H, -) & \cong & \prod_{K \in \mathscr{C}} \mathrm{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}[N_H(K)\backslash H], -) \\ & \cong & \prod_{K \in \mathscr{C}} H^n(N_H(K), -), \end{array}$$

where the last isomorphism follows from the Eckmann–Shapiro Lemma. Therefore, if $H^n(H, -)$ is finitary, it follows from Lemma 2.3 that $H^n(N_H(E), -)$ is also finitary. Hence, as H has cohomology almost everywhere finitary, we conclude that $N_H(E)$ also has cohomology almost everywhere finitary.

Now, as E is a finite group,

$$|N_H(E) : C_H(E)| < \infty,$$

and so by Lemma 2.1 we see that

$$C_H(E) \cong E \times C_N(E)$$

has cohomology almost everywhere finitary. It then follows from Proposition 2.4 that $C_N(E)$ is finitely generated, and hence polycyclic-by-finite.

Now, as $E \leq F$, it follows that $C_N(F) \leq C_N(E)$ and as every subgroup of a polycyclic-by-finite group is finitely generated, we see that $C_N(F)$ is finitely generated.

Finally, as N is a subgroup of G of finite index, it follows that

$$|C_G(F) : C_N(F)| < \infty,$$

and so $C_G(F)$ is finitely generated. Hence $N_G(F)$ is finitely generated, as required.

□

In the remainder of this section we shall prove the converse. Firstly, we need the following definition from [11]:

Definition 3.3. Let G be a group, and let $\Lambda(G)$ denote the poset of the non-trivial finite subgroups of G. We can view this poset as a G-simplicial complex, which we shall denote by $|\Lambda(G)|$, by the following method: An n-simplex in $|\Lambda(G)|$ is determined by each strictly increasing chain

$$H_0 < H_1 < \cdots < H_n$$

of $n+1$ non-trivial finite subgroups of G. The action of G on the set of non-trivial finite subgroups induces an action of G on $|\Lambda(G)|$, so that the stabilizer of a simplex is an intersection of normalizers; in the case of the simplex determined by the chain of subgroups above, the stabilizer is

$$\bigcap_{i=0}^{n} N_G(H_i).$$

This complex has the property that, for any non-trivial finite subgroup K of G, the K-fixed point complex $|\Lambda(G)|^K$ is contractible (for a proof of this, see Lemma 2.1 in [11]).

Next, we need the following two results of Kropholler and Mislin:

Proposition 3.4. *Let Y be a G-CW-complex of finite dimension n. Then Y can be embedded into an n-dimensional G-CW-complex $\widetilde{Y}$ which is $(n-1)$-connected in such a way that G acts freely outside Y.*

Proof. This is Lemma 4.4 of [11]. We can take $\widetilde{Y}$ to be the n-skeleton of the join

$$Y * \underbrace{G * \cdots * G}_{n}.$$

□

Proposition 3.5. *Let Y be an n-dimensional G-CW-complex which is $(n-1)$-connected, for some $n \geq 0$. Suppose that Y^K is contractible for all non-trivial finite subgroups K of G. Then the nth reduced homology group $\widetilde{H_n}(Y)$ is projective as a $\mathbb{Z}K$-module for all finite subgroups K of G.*

Proof. This is Proposition 6.2 of [11]. □

Finally, we require the following two lemmas:

Lemma 3.6. *Let*

$$0 \to F_1 \to F \to F_2 \to 0$$

be an exact sequence of functors from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$. If F_1 and F_2 are finitary, then so is F.

Proof. Let (M_λ) be a filtered colimit system of R-modules. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_\lambda F_1(M_\lambda) \arrow[r] \arrow[d, "f_1"] & \varinjlim_\lambda F(M_\lambda) \arrow[r] \arrow[d, "f"] & \varinjlim_\lambda F_2(M_\lambda) \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & F_1(\varinjlim_\lambda M_\lambda) \arrow[r] & F(\varinjlim_\lambda M_\lambda) \arrow[r] & F_2(\varinjlim_\lambda M_\lambda) \arrow[r] & 0
\end{tikzcd}$$

Now, as F_1 and F_2 are finitary, the maps f_1 and f_2 are isomorphisms. It then follows from the Five Lemma that f is an isomorphism, and we conclude that F is finitary.

□

Lemma 3.7. *Let G be a group. If we have an exact sequence of $\mathbb{Z}G$-modules*

$$0 \to A_r \to A_{r-1} \to \cdots \to A_0 \to \mathbb{Z} \to 0$$

such that, for each $i = 0, \ldots, r$, the functor $\operatorname{Ext}^*_{\mathbb{Z}G}(A_i, -)$ is finitary in all sufficiently high dimensions, then G has cohomology almost everywhere finitary.

Proof. If $r = 0$, then the result follows immediately. Assume, therefore, that $r \geq 1$, and proceed by induction.

If $r = 1$, then we have the short exact sequence

$$A_1 \rightarrowtail A_0 \twoheadrightarrow \mathbb{Z}$$

which gives the following long exact sequence:

$$\cdots \to \operatorname{Ext}^j_{\mathbb{Z}G}(A_0, -) \to \operatorname{Ext}^j_{\mathbb{Z}G}(A_1, -) \to H^{j+1}(G, -) \to$$
$$\to \operatorname{Ext}^{j+1}_{\mathbb{Z}G}(A_0, -) \to \operatorname{Ext}^{j+1}_{\mathbb{Z}G}(A_1, -) \to \cdots$$

and as both $\mathrm{Ext}^*_{\mathbb{Z}G}(A_0, -)$ and $\mathrm{Ext}^*_{\mathbb{Z}G}(A_1, -)$ are finitary in all sufficiently high dimensions, it follows from the Five Lemma that G has cohomology almost everywhere finitary.

Now suppose that we have shown this for $r-1$, and that we have an exact sequence

$$0 \to A_r \to A_{r-1} \to \cdots \to A_0 \to \mathbb{Z} \to 0$$

such that, for each $i = 0, \ldots, r$, the functor $\mathrm{Ext}^*_{\mathbb{Z}G}(A_i, -)$ is finitary in all sufficiently high dimensions. Let $K := \mathrm{Ker}(A_{r-2} \to A_{r-3})$, so we have the short exact sequence

$$A_r \rightarrowtail A_{r-1} \twoheadrightarrow K,$$

and an argument similar to above shows that $\mathrm{Ext}^*_{\mathbb{Z}G}(K, -)$ is finitary in all sufficiently high dimensions. We then have the following exact sequence:

$$0 \to K \to A_{r-2} \to \cdots \to A_0 \to \mathbb{Z} \to 0,$$

and the result now follows by induction.

□

Finally, we can now prove the converse:

Theorem 3.8. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. If the normalizer of every non-trivial finite subgroup of G is finitely generated, then G has cohomology almost everywhere finitary.*

Proof. Let $\Lambda(G)$ be the poset of all non-trivial finite subgroups of G, and let $|\Lambda(G)|$ denote its realization as a G-simplicial complex. As G has finite virtual cohomological dimension, there is a bound on the orders of its finite subgroups, and so $|\Lambda(G)|$ is finite-dimensional, say $\dim |\Lambda(G)| = r$. From Proposition 3.4, we can embed $|\Lambda(G)|$ into an r-dimensional G-CW-complex Y which is $(r-1)$-connected, such that G acts freely outside $|\Lambda(G)|$. Consider the augmented cellular chain complex of Y. As Y is $(r-1)$-connected, it has trivial homology except in dimension r, and so we have the following exact sequence:

$$0 \to \widetilde{H}_r(Y) \to C_r(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0.$$

In order to show that G has cohomology almost everywhere finitary, it is enough by Lemma 3.7 to show that the functors $\mathrm{Ext}^*_{\mathbb{Z}G}(\widetilde{H}_r(Y), -)$ and $\mathrm{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$, $0 \le l \le r$, are finitary in all sufficiently high dimensions.

Firstly, notice that for every non-trivial finite subgroup K of G, $Y^K = |\Lambda(G)|^K$, as the copies of G that we have added in the construction of Y have free orbits, and so have no fixed points under K. Thus,

Y is an r-dimensional G-CW-complex which is $(r-1)$-connected, such that Y^K is contractible for every non-trivial finite subgroup K of G. It then follows from Proposition 3.5 that $\widetilde{H}_r(Y)$ is projective as a $\mathbb{Z}K$-module for all finite subgroups K of G. Then by Proposition 3.1, $\widetilde{H}_r(Y)$ has finite projective dimension over $\mathbb{Z}G$, and so $\mathrm{Ext}^n_{\mathbb{Z}G}(\widetilde{H}_r(Y), -) = 0$, and thus is finitary, for all sufficiently large n.

Next, for each $0 \leq l \leq r$, consider the functor $\mathrm{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$. Provided that $n \geq 1$, we see that

$$\mathrm{Ext}^n_{\mathbb{Z}G}(C_l(Y), -) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(C_l(|\Lambda(G)|), -)$$

as the copies of G that we have added in the construction of Y have free orbits, and so the free-abelian group on them is a free module. Now,

$$\mathrm{Ext}^n_{\mathbb{Z}G}(C_l(|\Lambda(G)|), -) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}|\Lambda(G)|_l, -),$$

where $|\Lambda(G)|_l$ consists of all the l-simplicies

$$K_0 < K_1 < \cdots < K_l$$

in $|\Lambda(G)|$. As G acts on $|\Lambda(G)|_l$, we can therefore split $|\Lambda(G)|_l$ up into its G-orbits, where the stabilizer of such a simplex is $\bigcap_{i=0}^l N_G(K_i)$. We then obtain the following isomorphism:

$$\begin{aligned} \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}|\Lambda(G)|_l, -) &\cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}[\textstyle\coprod_{\mathscr{C}} \bigcap_{i=0}^l N_G(K_i)\backslash G], -) \\ &\cong \textstyle\prod_{\mathscr{C}} \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}[\bigcap_{i=0}^l N_G(K_i)\backslash G], -) \\ &\cong \textstyle\prod_{\mathscr{C}} H^n(\bigcap_{i=0}^l N_G(K_i), -), \end{aligned}$$

where the product is taken over a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial finite subgroups of G. Now, as G has finite virtual cohomological dimension, it follows from Lemma 1.2 that there are only finitely many conjugacy classes of finite subgroups, and so this product is finite.

Now, for each l-simplex $K_0 < \cdots < K_l$ we have

$$\bigcap_{i=0}^{l} N_G(K_i) \leq N_G(K_l).$$

Then, as $N_G(K_l)$ is finitely generated, it follows that $\bigcap_{i=0}^l N_G(K_i)$ is also finitely generated, and hence polycyclic-by-finite.

Therefore, $\bigcap_{i=0}^l N_G(K_i)$ is of type FP_∞, and so by Proposition 1.1, $H^n(\bigcap_{i=0}^l N_G(K_i), -)$ is finitary. Thus $\mathrm{Ext}^n_{\mathbb{Z}G}(C_l(Y), -)$ is isomorphic to a finite product of finitary functors, and hence by Lemma 3.6 is finitary. As this holds for all $n \geq 1$, we see that $\mathrm{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$ is finitary in all sufficiently high dimensions, which completes the proof. □

4. Proof of Corollary B

Corollary B. *Let G be a locally (polycyclic-by-finite) group. If G has cohomology almost everywhere finitary, then every subgroup of G also has cohomology almost everywhere finitary.*

Proof. As G has cohomology almost everywhere finitary, it follows from Theorem A that G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup of G is finitely generated.

Let H be any subgroup of G, so

$$\operatorname{vcd} H \leq \operatorname{vcd} G < \infty.$$

Also, let F be a non-trivial finite subgroup of H. Then $N_G(F)$ is finitely generated, hence polycyclic-by-finite, and as

$$N_H(F) \leq N_G(F),$$

we see that $N_H(F)$ is also finitely generated. Therefore, we conclude from Theorem A that H has cohomology almost everywhere finitary. □

This result does not hold in general, however, as the following proposition shows:

Proposition 4.1. *Let G be a group of type FP_∞ which has an infinitely generated subgroup H, and let Q be a non-trivial finite group. Then $G \times Q$ has cohomology almost everywhere finitary, but $H \times Q$ does not.*

Proof. As G is of type FP_∞, it follows that $G \times Q$ is also of type FP_∞, and so has cohomology almost everywhere finitary. However, as H is infinitely generated, it follows from Proposition 2.4 that $H^n(H \times Q, -)$ is not finitary for any n. □

Remark 4.2. Let G be the free group on two generators x, y, so G is of type FP_∞ (Example 2.6 in [2]), and let H be the subgroup of G generated by $y^n x y^{-n}$ for all n. We then have a counter-example showing that Corollary B does not hold in general.

5. A Result on Elementary Amenable Groups

Proposition C. *Let G be an elementary amenable group with cohomology almost everywhere finitary. Then G has finitely many conjugacy classes of finite subgroups, and $C_G(E)$ is finitely generated for every $E \leq G$ of order p.*

Proof. Let G be an elementary amenable group with cohomology almost everywhere finitary. Kropholler's Theorem (Theorem 2.1 in [10]) applies to a large class of groups, which includes all elementary amenable

groups. This theorem implies that G has a finite dimensional model for $\underline{E}G$, and that G has a bound on the orders of its finite subgroups. The proof of Lemma 1.2 generalizes immediately to the elementary amenable case, and we conclude that G has finitely many conjugacy classes of finite subgroups, and furthermore that G has finite virtual cohomological dimension. Therefore, we can choose a torsion-free normal subgroup N of G of finite index.

Let E be any subgroup of G of order p, and let $H := NE$. Following the proof of Theorem 3.2, we see that $N_H(E)$ has cohomology almost everywhere finitary. Hence,

$$C_H(E) \cong E \times C_N(E)$$

also has cohomology almost everywhere finitary, and so by Proposition 2.4 we see that $C_N(E)$ is finitely generated. The result now follows. □

6. Generalization to Groups of Finite Virtual Cohomological Dimension

In this section, we shall prove Theorem D. It suffices to show this for the case $R = \mathbb{F}_p$, by the following lemma:

Lemma 6.1. *Let G be a group, and R be a ring of prime characteristic p. Then $H^n(G, -)$ is finitary over R if and only if $H^n(G, -)$ is finitary over $\mathbb{F}_p$.*

Proof. If $H^n(G, -)$ is finitary over $\mathbb{F}_p$, then it follows from Lemma 2.2 that $H^n(G, -)$ is finitary over R.

Conversely, suppose that $H^n(G, -)$ is finitary over R; that is, the functor $\mathrm{Ext}^n_{RG}(R, -)$ is finitary. Let (M_λ) be a filtered colimit system of $\mathbb{F}_pG$-modules. Then $(M_\lambda \otimes_{\mathbb{F}_p} R)$ is a filtered colimit system of RG-modules, and so the natural map

$$\varinjlim_\lambda \mathrm{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} R) \to \mathrm{Ext}^n_{RG}(R, \varinjlim_\lambda M_\lambda \otimes_{\mathbb{F}_p} R)$$

is an isomorphism. Now, as an $\mathbb{F}_p$-vector space, $R \cong \mathbb{F}_p \oplus V$ for some $\mathbb{F}_p$-vector space V. Therefore, for each λ,

$$M_\lambda \otimes_{\mathbb{F}_p} R \cong M_\lambda \oplus M_\lambda \otimes_{\mathbb{F}_p} V,$$

and so

$$\mathrm{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} R) \cong \mathrm{Ext}^n_{RG}(R, M_\lambda) \oplus \mathrm{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} V).$$

It then follows from Lemma 2.3 that the natural map

$$\varinjlim_\lambda \mathrm{Ext}^n_{RG}(R, M_\lambda) \to \mathrm{Ext}^n_{RG}(R, \varinjlim_\lambda M_\lambda)$$

is an isomorphism. Now, we see from Chapter 0 of [2] that

$$\operatorname{Ext}^n_{RG}(R,-) \cong \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p,-)$$

on $\mathbb{F}_pG$-modules, so therefore it follows that the natural map

$$\varinjlim_{\lambda} \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p, M_\lambda) \to \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p, \varinjlim_{\lambda} M_\lambda)$$

is an isomorphism, and hence that $H^n(G,-)$ is finitary over $\mathbb{F}_p$. □

6.1. **Proof of (i) ⇒ (ii).**

Let G be a group of finite virtual cohomological dimension with cohomology almost everywhere finitary over $\mathbb{F}_p$. We begin by showing that the normalizer of every non-trivial elementary abelian p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$. In fact, we shall show that the normalizer of every non-trivial finite p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$.

Lemma 6.2. *Let N be any group, and Q be a non-trivial finite group whose order is divisible by p. If $N \times Q$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, then $N \times Q$ is of type FP_∞ over $\mathbb{F}_p$.*

Proof. Suppose that $N \times Q$ is not of type FP_∞ over $\mathbb{F}_p$, so N is not of type FP_∞ over $\mathbb{F}_p$. Therefore, there is some n such that $H^n(N,-)$ is not finitary over $\mathbb{F}_p$.

Let E be a subgroup of Q of order p, so by an argument similar to the proof of Proposition 2.4 we obtain, for each m, the following isomorphism of functors:

$$H^m(N \times E, -) \cong \bigoplus_{i=0}^{m} H^i(N,-),$$

for modules on which E acts trivially.

As $H^n(N,-)$ is not finitary over $\mathbb{F}_p$, it follows from Lemma 2.3 that $H^m(N \times E,-)$ is not finitary over $\mathbb{F}_p$ for all $m \geq n$. Therefore, by an easy generalization of Lemma 2.1, we see that $H^m(N \times Q,-)$ is not finitary over $\mathbb{F}_p$ for all $m \geq n$, which is a contradiction. □

Lemma 6.3. *Let G be a group of finite virtual cohomological dimension with cohomology almost everywhere finitary over $\mathbb{F}_p$, and let E be a subgroup of order p. Then the normalizer $N_G(E)$ of E is of type FP_∞ over $\mathbb{F}_p$.*

Proof. As G has finite virtual cohomological dimension, we can choose a torsion-free normal subgroup N of finite index. Let $H := NE$. A slight variation on the proof of Theorem 3.2 shows that $N_H(E)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$. Therefore,

$$C_H(E) \cong E \times C_N(E)$$

has cohomology almost everywhere finitary over $\mathbb{F}_p$, and by Lemma 6.2, we see that $C_H(E)$ is of type FP_∞ over $\mathbb{F}_p$. Thus, $N_G(E)$ is of type FP_∞ over $\mathbb{F}_p$, as required.

□

Theorem 6.4. *Let G be a group of finite virtual cohomological dimension with cohomology almost everywhere finitary over $\mathbb{F}_p$, and let F be a non-trivial finite p-subgroup. Then the normalizer $N_G(F)$ of F is of type FP_∞ over $\mathbb{F}_p$.*

Proof. Suppose that F has order p^k, where $k \geq 1$. We proceed by induction on k.

If $k = 1$, then the result follows from Lemma 6.3.

Suppose now that $k \geq 2$. As the centre $\zeta(F)$ of F is non-trivial, we can choose a subgroup $E \leq \zeta(F)$ of order p. Then $C_G(E)$ is of type FP_∞ over $\mathbb{F}_p$ by Lemma 6.3, and Proposition 2.7 in [2] shows that $C_G(E)/E$ is also of type FP_∞ over $\mathbb{F}_p$. By induction, the normalizer of F/E in $C_G(E)/E$, which is

$$(N_G(F) \cap C_G(E))/E,$$

is of type FP_∞ over $\mathbb{F}_p$. Another application of Proposition 2.7 in [2] shows that $N_G(F) \cap C_G(E)$ is of type FP_∞ over $\mathbb{F}_p$, and as

$$C_G(F) \leq N_G(F) \cap C_G(E) \leq N_G(F),$$

we see that $N_G(F)$ is of type FP_∞ over $\mathbb{F}_p$.

□

Next, we shall show that G has finitely many conjugacy classes of elementary abelian p-subgroups. Firstly, we need the following lemma:

Lemma 6.5. *Let G be a group. If $H^n(G, -)$ is finitary over $\mathbb{F}_p$, then $H^n(G, \mathbb{F}_p)$ is finite-dimensional as an $\mathbb{F}_p$-vector space.*

Proof. Suppose that $H^n(G, \mathbb{F}_p)$ is infinite-dimensional as an $\mathbb{F}_p$-vector space. By the Universal Coefficient Theorem, we have the following isomorphism:

$$H^n(G, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(H_n(G, \mathbb{F}_p), \mathbb{F}_p).$$

Hence $H_n(G, \mathbb{F}_p)$ is also infinite-dimensional as an $\mathbb{F}_p$-vector space, with basis $\{e_i : i \in I\}$, say. We then have:

$$H^n(G, \mathbb{F}_p) \cong \prod_I \mathbb{F}_p.$$

Next, let $\bigoplus_J \mathbb{F}_p$ be an infinite direct sum of copies of $\mathbb{F}_p$. As $H^n(G,-)$ is finitary over $\mathbb{F}_p$, the natural map

$$\bigoplus_J H^n(G,\mathbb{F}_p) \to H^n(G,\bigoplus_J \mathbb{F}_p)$$

is an isomorphism; that is,

$$\bigoplus_J \prod_I \mathbb{F}_p \cong \prod_I \bigoplus_J \mathbb{F}_p,$$

which is clearly a contradiction. □

Next, recall the following definition from [8]:

Definition 6.6. A homomorphism $\phi : A \to B$ of $\mathbb{F}_p$-algebras is called a *uniform F-isomorphism* if and only if there exists a natural number n such that:

- If $x \in \operatorname{Ker}\phi$, then $x^{p^n} = 0$; and
- If $y \in B$, then y^{p^n} is in the image of ϕ.

We also have the following result of Henn (Theorem A.4 in [8]):

Proposition 6.7. *If G is a discrete group such that there exists a finite-dimensional contractible G-CW-complex X with all cell stabilizers finite of bounded order, then there exists a uniform F-isomorphism*

$$\phi : H^*(G,\mathbb{F}_p) \to \lim_{\mathcal{A}_p(G)^{\mathrm{op}}} H^*(E,\mathbb{F}_p),$$

where $\mathcal{A}_p(G)$ denotes the category with objects the elementary abelian p-subgroups E of G, and morphisms the group homomorphisms which can be induced by conjugation by an element of G.

Finally, we can prove the following proposition, which is a generalization of a result of Henn (Theorem A.8 in [8]):

Proposition 6.8. *Let G be a group of finite virtual cohomological dimension with cohomology almost everywhere finitary over $\mathbb{F}_p$. Then G has finitely many conjugacy classes of elementary abelian p-subgroups.*

Proof. As G has finite virtual cohomological dimension, there is a finite dimensional model, say X, for the classifying space $\underline{E}G$ for proper actions (Exercise §VIII.3 in [5]). Thus, X is a finite dimensional contractible G-CW-complex with all cell stabilizers finite of bounded order, so it follows from Proposition 6.7 that there is a uniform F-isomorphism

$$\phi : H^*(G,\mathbb{F}_p) \to \lim_{\mathcal{A}_p(G)^{\mathrm{op}}} H^*(E,\mathbb{F}_p).$$

Now assume that there are infinitely many conjugacy classes of elementary abelian p-subgroups of G. As the order of the finite subgroups is

bounded, this means that there must be infinitely many maximal elementary abelian p-subgroups of G of the same rank k (although k itself need not necessarily be maximal). Following Henn's argument, we can use this fact to construct infinitely many linearly independent non-nilpotent classes in the inverse limit in some degree (for the details, see the proof of Theorem A.8 in [8]). Now, raising these to a large enough power and using the fact that ϕ is a uniform F-isomorphism, we see that $H^*(G, \mathbb{F}_p)$ is infinite-dimensional as an $\mathbb{F}_p$-vector space in some degree m such that $H^m(G, -)$ is finitary over $\mathbb{F}_p$. This gives a contradiction to Lemma 6.5.

□

6.2. Proof of (ii) ⇒ (iii).

This is immediate.

6.3. Proof of (iii) ⇒ (i).

Let G be a group of finite virtual cohomological dimension, such that G has finitely many conjugacy classes of elementary abelian p-subgroups and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over $\mathbb{F}_p$. We shall show that G has cohomology almost everywhere finitary over $\mathbb{F}_p$.

Firstly, let $\mathcal{A}_p(G)$ denote the poset of all the non-trivial elementary abelian p-subgroups of G, and let $\mathcal{S}_p(G)$ denote the poset of all the non-trivial finite p-subgroups of G. We see from Remark 2.3(i) in [17] that the inclusion of posets $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ induces a G-homotopy equivalence

$$|\mathcal{A}_p(G)| \simeq_G |\mathcal{S}_p(G)|$$

between the G-simplicial complexes.

Next, we need the following result (Proposition 2.7 §II in [3]):

Proposition 6.9. *Let X and Y be G-CW-complexes, and $\phi : X \to Y$ be a G-equivariant cellular map. Then ϕ is a G-homotopy equivalence if and only if $\phi^H : X^H \to Y^H$ is a homotopy equivalence for all subgroups H of G.*

We can now prove the following key lemma:

Lemma 6.10. *The complex $|\mathcal{A}_p(G)|^E$ is contractible for all $E \in \mathcal{A}_p(G)$.*

Proof. We follow an argument similar to the proof of Lemma 2.1 in [11]:

If $H \in \mathcal{S}_p(G)^E$, then EH is a p-subgroup of G. We can therefore define a function

$$f : \mathcal{S}_p(G)^E \to \mathcal{S}_p(G)^E$$

by $f(H) = EH$, so for all $H \in \mathcal{S}_p(G)^E$ we have:

$$H \leq f(H) \geq E.$$

We then see that $\mathcal{S}_p(G)^E$ is conically contractible in the sense of Quillen (see §1.5 in [15]), which implies that $|\mathcal{S}_p(G)^E|$ is contractible by Quillen's argument. Finally, by Proposition 6.9, we see that

$$|\mathcal{A}_p(G)|^E \simeq |\mathcal{S}_p(G)|^E = |\mathcal{S}_p(G)^E|,$$

and the result now follows.

□

Finally, we can now prove the following:

Theorem 6.11. *Let G be a group of finite virtual cohomological dimension. If G has finitely many conjugacy classes of elementary abelian p-subgroups, and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over $\mathbb{F}_p$, then G has cohomology almost everywhere finitary over $\mathbb{F}_p$.*

Proof. Let $\mathcal{A}_p(G)$ be the poset of all non-trivial elementary abelian p-subgroups of G, and let $|\mathcal{A}_p(G)|$ denote its realization as a G-simplicial complex. As G has finitely many conjugacy classes of elementary abelian p-subgroups, there must be a bound on their orders, and so $|\mathcal{A}_p(G)|$ is finite-dimensional, say $\dim |\mathcal{A}_p(G)| = r$. By Proposition 3.4, we can embed $|\mathcal{A}_p(G)|$ into an r-dimensional G-CW-complex Y which is $(r-1)$-connected, such that G acts freely outside $|\mathcal{A}_p(G)|$. The augmented cellular chain complex of Y then gives the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to \widetilde{H}_r(Y) \to C_r(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0,$$

which gives the following exact sequence of $\mathbb{F}_pG$-modules:

$$0 \to \widetilde{H}_r(Y) \otimes \mathbb{F}_p \to C_r(Y) \otimes \mathbb{F}_p \to \cdots \to C_0(Y) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0.$$

In order to show that G has cohomology almost everywhere finitary over $\mathbb{F}_p$, it is enough by an easy generalization of Lemma 3.7 to show that the functors $\mathrm{Ext}^*_{\mathbb{F}_pG}(\widetilde{H}_r(Y) \otimes \mathbb{F}_p, -)$ and $\mathrm{Ext}^*_{\mathbb{F}_pG}(C_l(Y) \otimes \mathbb{F}_p, -)$, $0 \leq l \leq r$, are finitary in all sufficiently high dimensions.

Firstly, notice that for every $E \in \mathcal{A}_p(G)$, $Y^E = |\mathcal{A}_p(G)|^E$, and hence is contractible, as the copies of G we have added in the construction of Y have free orbits, and so have no fixed points under E. Therefore, an easy generalization of Proposition 3.5 shows that $\widetilde{H}_r(Y) \otimes \mathbb{F}_p$ is projective as an $\mathbb{F}_pE$-module for all elementary abelian p-subgroups E of G.

Let K be a finite subgroup of G, so $\widetilde{H}_r(Y)\otimes\mathbb{F}_p$ restricted to K is an $\mathbb{F}_pK$-module with the property that its restriction to every elementary abelian p-subgroup of K is projective. It then follows from Chouinard's Theorem [6] that $\widetilde{H}_r(Y)\otimes\mathbb{F}_p$ is projective as an $\mathbb{F}_pK$-module. As this holds for every finite subgroup K of G, it then follows from Proposition 3.1 that $\widetilde{H}_r(Y)\otimes\mathbb{F}_p$ has finite projective dimension over $\mathbb{F}_pG$. Hence $\mathrm{Ext}^n_{\mathbb{F}_pG}(\widetilde{H}_r(Y)\otimes\mathbb{F}_p,-)=0$, and thus is finitary, for all sufficiently large n.

Next, for each $0\le l\le r$, consider the functor $\mathrm{Ext}^*_{\mathbb{F}_pG}(C_l(Y)\otimes\mathbb{F}_p,-)$. Provided that $n\ge 1$, we see that

$$\begin{aligned}\mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(Y)\otimes\mathbb{F}_p,-) &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(|\mathcal{A}_p(G)|)\otimes\mathbb{F}_p,-)\\ &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p|\mathcal{A}_p(G)|_l,-),\end{aligned}$$

where $|\mathcal{A}_p(G)|_l$ consists of all the l-simplicies

$$E_0<E_1<\cdots<E_l$$

in $|\mathcal{A}_p(G)|$. As G acts on $|\mathcal{A}_p(G)|_l$, we can therefore split $|\mathcal{A}_p(G)|_l$ up into its G-orbits, where the stabilizer of such a simplex is $\bigcap_{i=0}^l N_G(E_i)$. We then obtain the following isomorphism:

$$\begin{aligned}\mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p|\mathcal{A}_p(G)|_l,-) &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p[\textstyle\coprod_{\mathscr{C}}\bigcap_{i=0}^l N_G(E_i)\backslash G],-)\\ &\cong \textstyle\prod_{\mathscr{C}}\mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p[\bigcap_{i=0}^l N_G(E_i)\backslash G],-)\\ &\cong \textstyle\prod_{\mathscr{C}} H^n(\bigcap_{i=0}^l N_G(E_i),-),\end{aligned}$$

where the product is taken over a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial elementary abelian p-subgroups of G. As we are assuming that G has only finitely many such conjugacy classes, this product is finite.

Now, for each l-simplex $E_0<E_1<\cdots<E_l$ we have

$$C_G(E_l)\le\bigcap_{i=0}^l N_G(E_i)\le N_G(E_l),$$

and so

$$|N_G(E_l):\bigcap_{i=0}^l N_G(E_i)|<\infty.$$

Then, as $N_G(E_l)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, we see from an easy generalization of Lemma 2.1 that $\bigcap_{i=0}^l N_G(E_i)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, and so for all sufficiently large n, $H^n(\bigcap_{i=0}^l N_G(E_i),-)$ is finitary over $\mathbb{F}_p$. Therefore, for all sufficiently large n, $\mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(Y)\otimes\mathbb{F}_p,-)$ is isomorphic to a finite

product of finitary functors, and hence is finitary, which completes the proof. ☐

7. A Question of Leary and Nucinkis

In [13], Leary and Nucinkis posed the following question: If G is a group of type VFP over $\mathbb{F}_p$, and P is a p-subgroup of G, is the centralizer $C_G(P)$ of P necessarily of type VFP over $\mathbb{F}_p$?

In this section, we shall give a positive answer to this question. Firstly, recall (see §2 of [13]) that a group G is said to be of type VFP over $\mathbb{F}_p$ if and only if it has a subgroup of finite index which is of type FP over $\mathbb{F}_p$.

Proposition 7.1. *Let G be a group which has a subgroup H of finite index with $\operatorname{cd}_{\mathbb{F}_p} H < \infty$. Then there exists a finite dimensional G-CW-complex X with finite cell stabilizers such that*

$$0 \to C_r(X) \otimes \mathbb{F}_p \to \cdots \to C_0(X) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0$$

is an exact sequence of $\mathbb{F}_pG$-modules.

Proof. As $\operatorname{cd}_{\mathbb{F}_p} H < \infty$, it follows from an easy generalization of Theorem 7.1 §VIII in [5] that there exists a finite dimensional free H-CW-complex X' with the property that $\widetilde{C}_*(X') \otimes \mathbb{F}_p$ is exact. Set

$$X := \operatorname{Hom}_H(G, X').$$

An easy generalization of the proof of Theorem 3.1 §VIII in [5] then shows that X has the required properties. ☐

Next, we prove the following key lemma, which is a variation on Proposition 3.1:

Lemma 7.2. *Let G be a group of type* VFP *over $\mathbb{F}_p$, and M be an $\mathbb{F}_pG$-module. If M is projective as an $\mathbb{F}_pK$-module for all finite subgroups K of G, then M has finite projective dimension over $\mathbb{F}_pG$.*

Proof. As G is of type VFP over $\mathbb{F}_p$, we see from Proposition 7.1 that there exists a finite-dimensional G-CW-complex X with finite cell stabilizers, such that

$$0 \to C_r(X) \otimes \mathbb{F}_p \to \cdots \to C_0(X) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0$$

is exact. Now, for each k, $C_k(X)$ is a permutation module,

$$C_k(X) \cong \bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[G_\sigma \backslash G],$$

where Σ_k is a set of G-orbit representatives of k-cells in X, and G_σ is the stabilizer of σ. If we tensor the above exact sequence with M, then we obtain the following:

$$0 \to M \otimes_{\mathbb{F}_p} (C_r(X) \otimes \mathbb{F}_p) \to \cdots \to M \otimes_{\mathbb{F}_p} (C_0(X) \otimes \mathbb{F}_p) \to M \to 0,$$

where, for each k, we have

$$M \otimes_{\mathbb{F}_p} (C_k(X) \otimes \mathbb{F}_p) \cong \bigoplus_{\sigma \in \Sigma_k} M \otimes_{\mathbb{F}_p G_\sigma} \mathbb{F}_p G.$$

Now, as M is projective as an $\mathbb{F}_p G_\sigma$-module, we see that $M \otimes_{\mathbb{F}_p G_\sigma} \mathbb{F}_p G$ is projective as an $\mathbb{F}_p G$-module. Therefore, $M \otimes_{\mathbb{F}_p} (C_k(X) \otimes \mathbb{F}_p)$ is projective as an $\mathbb{F}_p G$-module, and so the above exact sequence is a projective resolution of M, and we then conclude that M has finite projective dimension over $\mathbb{F}_p G$. □

We can now answer Leary and Nucinkis' question in the case where P has order p. This is a variation on Lemma 6.3:

Proposition 7.3. *Let G be a group of type* VFP *over $\mathbb{F}_p$, and let P be a subgroup of G of order p. Then $C_G(P)$ is of type* VFP *over $\mathbb{F}_p$.*

Proof. As G is of type VFP over $\mathbb{F}_p$, we can choose a normal subgroup N of finite index which is of type FP over $\mathbb{F}_p$. Let $H := NP$, so H is of type VFP over $\mathbb{F}_p$.

Next, let $\mathcal{A}_p(H)$ denote the set of all non-trivial elementary abelian p-subgroups of H, so $\mathcal{A}_p(H)$ consists of subgroups of order p. Now H acts on this set by conjugation, so the stabilizer of any $E \in \mathcal{A}_p(H)$ is simply $N_H(E)$. Also, for each $E \in \mathcal{A}_p(H)$, we see that the set of E-fixed points $\mathcal{A}_p(H)^E$ is simply the set $\{E\}$. We then have the following short exact sequence of $\mathbb{F}_p H$-modules:

$$J \rightarrowtail \mathbb{F}_p \mathcal{A}_p(H) \overset{\varepsilon}{\twoheadrightarrow} \mathbb{F}_p,$$

where ε denotes the augmentation map, and we see that for each $E \in \mathcal{A}_p(H)$, J is free as an $\mathbb{F}_p E$-module with basis $\{E' - E : E' \in \mathcal{A}_p(H)\}$. Therefore, if K is any finite subgroup of H, we see that J restricted to K is an $\mathbb{F}_p K$-module such that its restriction to every elementary abelian p-subgroup of K is free. It then follows from Chouinard's Theorem [6] that J is projective as an $\mathbb{F}_p K$-module. As this holds for every finite subgroup K of H, it follows from Lemma 7.2 that J has finite projective dimension over $\mathbb{F}_p H$.

An argument similar to the proof of Theorem 3.2 then shows that $N_H(P)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$. Hence,

$$C_H(P) \cong P \times C_N(P)$$

has cohomology almost everywhere finitary over $\mathbb{F}_p$, and by Lemma 6.2, $C_N(P)$ is of type FP_∞ over $\mathbb{F}_p$. Finally, as

$$\mathrm{cd}_{\mathbb{F}_p} C_N(P) \leq \mathrm{cd}_{\mathbb{F}_p} N < \infty,$$

we see that $C_N(P)$ is of type FP over $\mathbb{F}_p$, and the result now follows. □

We can now answer Leary and Nucinkis' question. This is a variation on Theorem 6.4:

Theorem E. *Let G be a group of type* VFP *over $\mathbb{F}_p$, and P be a p-subgroup of G. Then the centralizer $C_G(P)$ of P is also of type* VFP *over $\mathbb{F}_p$.*

Proof. If P is trivial, then the result is immediate. Assume, therefore, that P has order p^k, where $k \geq 1$. We proceed by induction on k:

If $k = 1$, then the result follows from Proposition 7.3.

Suppose now that $k \geq 2$. Choose a subgroup $E \leq \zeta(P)$ of order p. Then $C_G(E)$ is of type VFP over $\mathbb{F}_p$ by Proposition 7.3, and so $C_G(E)$ has a normal subgroup N of finite index which is of type FP over $\mathbb{F}_p$. Let $H := NE$. Then $C_G(E)/E$ has the subgroup H/E of finite index, with $H/E \cong N$ of type FP over $\mathbb{F}_p$, and so $C_G(E)/E$ is of type VFP over $\mathbb{F}_p$. By induction, the centralizer of P/E in $C_G(E)/E$ is of type VFP over $\mathbb{F}_p$. Hence the normalizer of P/E in $C_G(E)/E$, which is

$$(N_G(P) \cap C_G(E))/E,$$

is of type FP_∞ over $\mathbb{F}_p$, and by Proposition 2.7 in [2] we see that $N_G(P) \cap C_G(E)$ is of type FP_∞ over $\mathbb{F}_p$. Then, as

$$C_G(P) \leq N_G(P) \cap C_G(E) \leq N_G(P),$$

we conclude that $C_G(P)$ is also of type FP_∞ over $\mathbb{F}_p$.

Now, as G is of type VFP over $\mathbb{F}_p$, it has a subgroup S of finite index which is of type FP over $\mathbb{F}_p$. Therefore, $C_S(P)$ is of type FP_∞ over $\mathbb{F}_p$, and as

$$\mathrm{cd}_{\mathbb{F}_p} C_S(P) \leq \mathrm{cd}_{\mathbb{F}_p} S < \infty,$$

we see that $C_S(P)$ is of type FP over $\mathbb{F}_p$, and the result now follows. □

8. Groups Possessing a Finite Dimensional Model for $\underline{E}G$

In this short section we consider groups possessing a finite dimensional model for the classifying space $\underline{E}G$ for proper actions. The proof of Theorem 3.8 generalizes immediately to give us the following:

Proposition F. *Let G be a group which possesses a finite dimensional model for the classifying space $\underline{E}G$ for proper actions. If*

(i) *G has finitely many conjugacy classes of finite subgroups; and*

(ii) *The normalizer of every non-trivial finite subgroup of G has cohomology almost everywhere finitary,*

Then G has cohomology almost everywhere finitary.

However, the converse is false. In fact, it is false even for the subclass of groups of finite virtual cohomological dimension, as we shall now show. We need the following result of Leary (Theorem 20 in [12]):

Proposition 8.1. *Let Q be a finite group not of prime power order. Then there is a group H of type* F *and a group $G = H \rtimes Q$ such that G contains infinitely many conjugacy classes of subgroups isomorphic to Q and finitely many conjugacy classes of other finite subgroups.*

As H is of type F, it has a finite Eilenberg–Mac Lane space, say Y. As the universal cover $\widetilde{Y}$ of Y is contractible, its augmented cellular chain complex is an exact sequence of $\mathbb{Z}H$-modules:

$$0 \to C_n(\widetilde{Y}) \to \cdots \to C_0(\widetilde{Y}) \to \mathbb{Z} \to 0,$$

and as Y has only finitely many cells in each dimension, we see that each $C_k(\widetilde{Y})$ is finitely generated.

Hence, we see that H has finite cohomological dimension, and is of type FP_∞. Therefore, G is a group of finite virtual cohomological dimension which is of type FP_∞, and hence has cohomology almost everywhere finitary, but G does *not* have finitely many conjugacy classes of finite subgroups, which gives us a counter-example to the converse of Proposition F above.

9. Proof of Lemma 1.2

9.1. **Proof of (i) $\Rightarrow$ (ii).**

Proposition 9.1. *Let G be a locally (polycyclic-by-finite) group such that there is a finite dimensional model for $\underline{E}G$ and there is a bound on the orders of the finite subgroups of G. Then G has finite virtual cohomological dimension.*

Proof. Let X be a finite dimensional model for $\underline{E}G$, and let $r = \dim X$. Then, for each k, $C_k(X)$ is a permutation module,

$$C_k(X) \cong \bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[G_\sigma \backslash G],$$

where Σ_k is a set of G-orbit representatives of k-cells in X, and G_σ is the stabilizer of σ, so in particular each G_σ is finite. Then

$$C_k(X) \otimes \mathbb{Q} \cong \bigoplus_{\sigma \in \Sigma_k} \mathbb{Q} \otimes_{\mathbb{Q}G_\sigma} \mathbb{Q}G,$$

and as $\mathbb{Q}G_\sigma$ is semisimple, $\mathbb{Q}$ is a projective $\mathbb{Q}G_\sigma$-module. Hence, $\mathbb{Q}\otimes_{\mathbb{Q}G_\sigma}\mathbb{Q}G$ is a projective $\mathbb{Q}G$-module, and so $C_k(X)\otimes\mathbb{Q}$ is a projective $\mathbb{Q}G$-module. We then have that

$$0 \to C_r(X) \otimes \mathbb{Q} \to \cdots \to C_0(X) \otimes \mathbb{Q} \to \mathbb{Q} \to 0$$

is a projective resolution of the trivial $\mathbb{Q}G$-module, and hence that G has finite rational cohomological dimension.

Now, according to Hillman and Linnell [9], the Hirsch length of an elementary amenable group is bounded above by its rational cohomological dimension, so we conclude that G has finite Hirsch length.

Next, let $\tau(G)$ denote the unique largest locally finite normal subgroup of G. As there is a bound on the orders of the finite subgroups of G, we see that $\tau(G)$ must be finite.

Then, as G is an elementary amenable group of finite Hirsch length, it follows from a result of Wehrfritz [18] that $G/\tau(G)$ has a poly-(torsion-free abelian) characteristic subgroup of finite index. Hence, G has a poly-(torsion-free abelian) characteristic subgroup, say S, of finite index. We see that S has finite Hirsch length, and hence finite cohomological dimension. We then conclude that G has finite virtual cohomological dimension, as required. □

9.2. **Proof of (ii) $\Rightarrow$ (iii).**

We begin by proving the following lemma:

Lemma 9.2. *Let Q be a finite group, and A be a $\mathbb{Z}$-torsion-free $\mathbb{Z}Q$-module of finite Hirsch length. Then $H^1(Q,A)$ is finite.*

Proof. As Q is finite, $H^1(Q,A)$ has exponent dividing the order of Q (Corollary 10.2 §III in [5]). We have the following short exact sequence:

$$A \overset{|Q|}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} A/|Q|A.$$

Passing to the long exact sequence in cohomology, we obtain the following monomorphism:

$$H^1(Q,A) \overset{\pi^*}{\rightarrowtail} H^1(Q, A/|Q|A).$$

Now, as $A/|Q|A$ has finite exponent and finite Hirsch length, it is finite. It then follows that $H^1(Q,A)$ is finite. □

Next, note that all groups of finite virtual cohomological dimension possess a finite dimensional model for $\underline{E}G$ (Exercise §VIII.3 in [5]), so it suffices to prove the following:

Proposition 9.3. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. Then G has finitely many conjugacy classes of finite subgroups.*

Proof. As G has finite virtual cohomological dimension, it must have a bound on the orders of its finite subgroups. Therefore, the same argument as in the previous subsection shows that G is a poly-(torsion-free abelian)-by-finite group of finite Hirsch length. We proceed by induction on the Hirsch length $h(G)$ of G.

If $h(G)$=1, then G has a torsion-free abelian normal subgroup A of finite Hirsch length such that $G/A = Q$ is finite. There is a 1-1 correspondence between the conjugacy classes of complements to A in G and $H^1(Q, A)$ (Result 11.1.3 in [16]). Therefore, by Lemma 9.2, we see that G has finitely many conjugacy classes of finite subgroups.

Suppose $h(G) > 1$. We know that G has a torsion-free abelian normal subgroup A of finite Hirsch length. As $h(G) > h(G/A)$, we see by induction that G/A has finitely many conjugacy classes of finite subgroups. Let F be a finite subgroup of G, so AF lies in one of finitely many conjugacy classes, say those represented by $AK_1, \ldots, AK_m$. Then, as each $H^1(K_i, A)$ is finite, there are only finitely many conjugacy classes of complements to A in AK_i, and F must lie in one of those. □

9.3. Proof of (iii) ⇒ (i).

Let G be a group with finitely many conjugacy classes of finite subgroups. Then it is clear that there must be a bound on the orders of its finite subgroups.

References

1. Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.
2. Robert Bieri, *Homological dimension of discrete groups*, second ed., Queen Mary College Mathematical Notes, Queen Mary College Department of Pure Mathematics, London, 1981.
3. Glen E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34, Springer-Verlag, Berlin, 1967.
4. Kenneth S. Brown, *Homological criteria for finiteness*, Comment. Math. Helv. **50** (1975), 129–135.
5. ______, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.
6. Leo G. Chouinard, *Projectivity and relative projectivity over group rings*, J. Pure Appl. Algebra **7** (1976), no. 3, 287–302.
7. Jonathan Cornick and Peter H. Kropholler, *Homological finiteness conditions for modules over strongly group-graded rings*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 1, 43–54.
8. Hans-Werner Henn, *Unstable modules over the Steenrod algebra and cohomology of groups*, **63** (1998), 277–300.

9. J. A. Hillman and P. A. Linnell, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable*, J. Austral. Math. Soc. Ser. A **52** (1992), no. 2, 237–241.
10. Peter H. Kropholler, *Groups with many finitary cohomology functors*, (Preprint, University of Glasgow 2007).
11. Peter H. Kropholler and Guido Mislin, *Groups acting on finite-dimensional spaces with finite stabilizers*, Comment. Math. Helv. **73** (1998), no. 1, 122–136.
12. Ian J. Leary, *On finite subgroups of groups of type VF*, Geom. Topol. **9** (2005), 1953–1976 (electronic).
13. Ian J. Leary and Brita E. A. Nucinkis, *Some groups of type VF*, Invent. Math. **151** (2003), no. 1, 135–165.
14. Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.
15. Daniel Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128.
16. Derek J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1993.
17. J. Thévenaz and P. J. Webb, *Homotopy equivalence of posets with a group action*, J. Combin. Theory Ser. A.
18. B. A. F. Wehrfritz, *On elementary amenable groups of finite Hirsch number*, J. Austral. Math. Soc. Ser. A **58** (1995), no. 2, 219–221.
19. Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, United Kingdom

E-mail address: `m.hamilton@maths.gla.ac.uk`